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A CLASS OF SEQUENTIAL PROCEDURES FOR CHOOSING ONE OF K HYPOTHESES CONCERNING THE UNKNOWN DRIFT PARAMETER OF THE WIENER PROCESS [1/]

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2/ Currently at Stanford University.

1. Summary and Introduction.

Let $X(t) \sim N(\mu t, \sigma^2 t)$ be a Wiener process for $t \geq 0$. Suppose σ^2 is known and that $H_1, \ldots, H_K$ are K hypotheses concerning the unknown drift parameter μ. A general sequential testing procedure presents itself in the following form: Continue to observe $X(t)$ until, for the first time s, the point $(s, X(s)) \epsilon B$, a subset of the right half plane. B will be called the boundary. Let B be partitioned into K disjoints subsets $B_i (i=1, \ldots, K)$ called boundary sets. Then, if $(s, X(s)) \epsilon B_i$, one accepts H_i.

In this paper, we will assume that each boundary set B_i is composed of a finite number of straight lines or line sgements called boundary lines. Any test using such a procedure (for some K) will be called a boundary test. We shall derive recursive methods for computing the exact O.C. functions, the average sample time (AST), and, in fact, all the moments of the sampling time τ.

Even though the study of tests for Wiener processes seems to be of intrinsic interest, their study usually has been motivated by other considerations. T.W. Anderson [1] has developed an approximate test for the unknown mean of the normal distribution by relating his testing procedure with a corresponding boundary test. In a similar manner, one may derive Wald's sequential probability ratio test, the Neyman-Pearson fixed sample size test, and multistage tests, for the mean of the normal distribution.

The restriction to a finite number of (straight) boundary lines may seem unpleasant but, of course, curvilinear boundaries can be approximated with polygonal lines. One consequence of the restriction is that there will exist at most a finite number of times at which an end-point of a boundary line, an intersection of two boundary lines, or a vertical boundary line occurs. These times, coupled with times $t = 0$ and $t = \infty$, will be called critical times. The recursive methods developed below will allow us to reduce the computations of the O.C. functions and the moments of τ to the computation of certain "fundamental probabilities" involving events which occur between critical times. We shall denote the critical times as

$$0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = \infty .$$

The points (s,x), with $s \geq 0$, which are not in the boundary B will be denoted by B_0. A point in B_0 which can be reached by a continuous sample path before time τ will be called an accessible point. Otherwise, the point is an inaccessible point.

An important subclass of the class of boundary tests is the class of boundary tests which are closed for all values of μ. This class has three characterizations (See [3].):

(i) $P_\mu[\tau < \infty] = 1$ for all μ .

(ii) $E_\mu(\tau^r) < \infty$ for all μ, for $r = 1,2,\ldots$.

(iii) Every accessible point of the form (t_m,x) is contained between two parallel, non-vertical boundary lines extending to infinity. 1/

1/ In order to avoid trivial set-theoretic complications, it will be assumed, without loss of generality, that the boundary B is a closed set.

Sections two and three will present the recursive procedures for computing the O.C. functions and the moments of the stopping time τ, respectively. Section four discusses the computation of the "fundamental probabilities."

The reader will quickly perceive that the recursive methods presented below need not be limited to straight line boundaries. The limitation arises from the rather restricted class of computable fundamental probabilities (at the present time).

2. The O.C. Functions.

In what follows, we shall use the letter s (subscripted or not) to denote a fixed time and shall use the letter x (with the same subscript, if any) to denote the value assumed by the Wiener process $X(t)$ at time $t = s$. With these conventions, there should be no difficulty in distinguishing between a time interval (s_1,s_2) and a point (s_1,x_1). Also, denote the set $B-B_i$ by $\overline{B}_i$, for $i=1,\ldots,K$.

Let μ and σ^2 be fixed parameters. Define, for $0 \leq s_1 < s_2 \leq \infty$,

$$(2.1)\quad P_i(s_1,s_2;x_1) \equiv P[(t,X(t))\epsilon B_i \text{ before } \overline{B}_i \text{ for some time } t\epsilon(s_1,s_2]|X(s_1)=x_1],$$

for $i=1,\ldots,K$, and

$$(2.2)\quad P_0(s_1,s_2;x_1) \equiv P[(t,X(t))\epsilon B_0 \text{ for all times } t\epsilon(s_1,s_2]|X(s_1)=x_1].$$

Define, for $0 \leq s_1 < s_2 < \infty$,

$$(2.3)\quad Q_i(s_1,s_2;x_1,x_2) \equiv P[(t,X(t))\epsilon B_i \text{ before } \overline{B}_i \text{ for some time } t\epsilon(s_1,s_2]|X(s_j)=x_j,\ j=1,2],$$

for $i=1,\ldots,K$, and

$$(2.4)\quad Q_0(s_1,s_2;x_1,x_2) \equiv P[(t,X(t))\epsilon B_0 \text{ for all times } t\epsilon(s_1,s_2]|X(s_j)=x_j,\ j=1,2].$$

Finally, letting $N(x|a,b)$ denote the normal density with mean a and variance b, define, for $0 \leq s_1 < s_2 < s_3 < \infty$,

$$(2.5) \qquad g(x_2|s_1,s_2;x_1) \equiv N(x_2|x_1+(s_2-s_1)\mu, \sigma^2(s_2-s_1)) ,$$

the conditional density of $X(s_2)$ given $X(s_1) = x_1$, and

$$(2.6) \qquad h(x_2|s_1,s_2,s_3;x_1,x_3) \equiv N(x_2|(x_3-x_1) \cdot [\frac{s_2-s_1}{s_3-s_1}], \sigma^2(s_3-s_2) \cdot [\frac{s_2-s_1}{s_3-s_1}]),$$

the conditional density of $X(s_2)$ given $X(s_j) = x_j$, for $j=1,3$.

Now, we find that $P_i(0,\infty;0)$ is the probability of accepting hypothesis H_i for $i=1,\ldots,K$.[1] The test is closed if, and only if, $P_0(0,\infty;0) = 0$. By the expression "fundamental probability," we shall mean any one of the probabilities (2.1), (2.2), (2.3), or (2.4) for which the (open) interval (s_1,s_2) contains no critical times $(t_j, j=0,1,\ldots,m+1)$.

The basis for induction is the following trivial observation: If s_1, s_2, and s_3 are any three times, with $0 \leq s_1 < s_2 < s_3$, and s_2 is a critical time, then (s_1,s_3) contains more critical times than does either (s_1,s_2) or (s_2,s_3). The induction step is contained within these four recursive formulas:

$$(2.7) \qquad P_i(s_1,s_3;x_1) = P_i(s_1,s_2;x_1) + \int_{-\infty}^{\infty} Q_0(s_1,s_2;x_1,x_2)P_i(s_2,s_3;x_2) \cdot g(x_2|s_1,s_2;x_1)dx_2 ,$$

for $i=1,\ldots,K$.

$$(2.8) \qquad P_0(s_1,s_2;x_1) = \int_{-\infty}^{\infty} Q_0(s_1,s_2;x_1,x_2)P_0(s_2,s_3;x_2) \cdot g(x_2|s_1,s_2;x_1)dx_2 .$$

[1] We shall avoid a trivial contradiction by assuming that the point $(0,0)$ is an accessible point. (i.e., $(0,0) \notin B$.)

$$(2.9)\quad Q_i(s_1,s_3;x_1,x_3) = \int_{-\infty}^{\infty} [Q_i(s_1,s_2;x_1,x_2)+Q_0(s_1,s_2;x_1,x_2)\cdot Q_i(s_2,s_3;x_2,x_3)] \cdot h(x_2|s_1,s_2,s_3;x_1,x_3)dx_2 ,$$

for $i=1,\ldots,K$.

$$(2.10)\quad Q_0(s_1,s_3;x_1,x_3) = \int_{-\infty}^{\infty} Q_0(s_1,s_2;x_1,x_2)Q_0(s_2,s_3;x_2,x_3) \cdot h(x_2|s_1,s_2,s_3;x_1,x_3)dx_2 .$$

Formulas (2.7) through (2.10) follow from rather elementary arguments involving conditional probabilities.

Computing the O.C. functions:

One simple induction procedure presents itself.

Procedure I:

Compute successively, for $s = t_1,\ldots,t_m$,

(i) $Q_0(0,s;0,x)$ for $x\epsilon(-\infty,\infty)$,

(ii) $P_i(0,s;0)$ for $i=1,\ldots,K$,

and then compute

(iii) $P_i(0,\infty;0)$ for $i=1,\ldots,K$.

This procedure uses formulas (2.8) and (2.10). If the test is truncated at time t_m, step (iii) is not needed. In fact, whenever $Q_0(0,s;0,x) = 0$, we do not need to compute $P_i(s,s';x)$ for $s' > s$. That is, the value of $P_i(s,s';x)$, for $s' > s$, is not important unless (s,x) is an accessible point.

If T is any positive time, the conditional process $X(t)$ given $X(T)$ is independent of μ for $0 \leq t \leq T$. It follows that the Q's, unlike the P's, do not depend on the fixed value of μ. When one wishes to compute the O.C. functions for many values of μ, a faster procedure (for high speed machines) is suggested.

Procedure II:

Compute successively, for $s = t_1,\ldots,t_m$, (using (2.9) and (2.10))

(i) $Q_i(0,s;0,x)$ for $x\epsilon(-\infty,\infty)$ and $i=0,1,\ldots,K$,

and then compute

$$(2.11)\quad \text{(ii)}\quad P_i(0,\infty;0) = \int_{-\infty}^{\infty} [Q_i(0,t_m;0,x)+Q_0(0,t_m;0,x)P_i(t_m,\infty;x)] \cdot g(x|0,t_m;0)dx\ .$$

Equation (2.11) represents a trivial modification of (2.7). If the test is truncated at time t_m, (2.11) simplifies, and whenever the point (s,x) is not an accessible point, certain subsequent computations may be avoided.

If one desires computations for only a few values of μ or desires explicit analytic expressions for the O.C. functions, procedure I is preferable to procedure II as a rule.

3. Moments of τ.

In this section, we shall exploit the well-known formula

$$(3.1) \qquad E(\tau^r) = r \int_0^\infty t^{r-1} P[\tau > t]dt \text{ for } r=1,2,\ldots ,$$

which even holds for unclosed tests.

Define, for $0 \leq s_1 < s_2 \leq \infty$,

$$(3.2) \qquad U_i(s_1,s_2;x_1) \equiv \int_{s_1}^{s_2} s^i P_0(s_1,s;x_1)ds, \quad \text{for } i=0,1,\ldots .$$

Also, define, for $0 \leq s_1 < s_2 < \infty$,

$$(3.3) \qquad V_i(s_1,s_2;x_1,x_2) = \int_{s_1}^{s_2} s^i \int_{-\infty}^{\infty} Q_0(s_1,s;x_1,x)h(x|s_1,s,s_2;x_1,x_2)dxds ,$$

for $i=0,1,\ldots$.

Then

$$(3.4) \qquad E(\tau^r) = r\, U_{r-1}(0,\infty;0) \text{ for } r=1,2,\ldots .$$

If $s_1 \leq s \leq s_2$ and the interval (s_1,s_2) contains no critical times, $P_0(s_1,s;x_1)$ and $Q_0(s_1,s;x_1,x)$ are fundamental probabilities. Then $U_i(s_1,s_2,x_1)$ and $V_i(s_1,s_2,x_1,x_2)$ may be computed directly from (3.2) and (3.3), respectively.

The induction step is based on these two recursive formulas:

$$(3.5) \qquad U_i(s_1,s_3;x_1) = U_i(s_1,s_2;x_1) + \int_{-\infty}^{\infty} Q_0(s_1,s_2;x_1,x_2)U_i(s_2,s_3;x_2) \cdot g(x_2|s_1,s_2;x_1)dx_2 ,$$

for $0 \leq s_1 < s_2 < s_3 < \infty$ and $i=0,1,\ldots$.

$$(3.6)\quad V_i(s_1,s_3;x_1,x_3) = \int_{-\infty}^{\infty} [V_i(s_1,s_2;x_1,x_2)+Q_0(s_1,s_2;x_1,x_2)V_i(s_2,s_3;x_2,x_3)] \cdot h(x_2|s_1,s_2,s_3;x_1,x_3)dx_2 ,$$

for $0 \leq s_1 < s_2 < s_3 < \infty$ and $i=0,1,\ldots$.

Formula (3.5) holds for $s_3 = \infty$ if we adopt the convention of calling the integrand in (3.5) zero whenever $Q_0(s_1,s_2;x_1,x_2) = 0$ (i.e., $0 \cdot c \equiv 0$ for $c \leq \infty$). (3.5) may be seen from

$$\int_{-\infty}^{\infty} Q_0(s_1,s_2;x_1,x_2)U_i(s_2,s_3;x_2)g(x_2|s_1,s_2;x_1)dx_2$$

$$= \int_{s_2}^{s_3} s^i \int_{-\infty}^{\infty} Q_0(s_1,s_2;x_1,x_2)P_0(s_2,s;x_2)g(x_2|s_1,s_2;x_1)dx_2 ds$$

$$= \int_{s_2}^{s_3} s^i \; P_0(s_1,s;x_1)ds \ .$$

The verification of (3.6) is somewhat more delicate. It suffices to show that

$$(3.7)\quad \int_{-\infty}^{\infty} V_i(s_1,s_2;x_1,x_2)h(x_2|s_1,s_2,s_3;x_1,x_3)dx_2$$

$$= \int_{s_1}^{s_2} s^i \int_{-\infty}^{\infty} Q_0(s_1,s;x_1,x)h(x|s_1,s,s_3;x_1,x_3)dxds$$

and

$$(3.8)\quad \int_{-\infty}^{\infty} Q_0(s_1,s_2;x_1,x_2)V_i(s_2,s_3;x_2,x_3)h(x_2|s_1,s_2,s_3;x_1,x_3)dx_2$$

$$= \int_{s_2}^{s_3} s^i \int_{-\infty}^{\infty} Q_0(s_1,s;x_1,x)h(x|s_1,s,s_3;x_1,x_3)dxds \ .$$

(3.7) follows from an interchange of integrals and application of the fact that

$$\int_{-\infty}^{\infty} h(x|s_1,s,s_2;x_1,x_2)h(x_2|s_1,s_2,s_3;x_1,x_3)dx_2 = h(x|s_1,s,s_3;x_1,x_3) \ ,$$

for $s_1 < s < s_2 < s_3$.

To show (3.8), one first observes that, for $s_1 < s_2 < s < s_3$,

$$(3.9) \quad h(x|s_2,s,s_3;x_2,x_3) \cdot h(x_2|s_1,s_2,s_3;x_1,x_3) = h(x|s_1,s,s_3;x_1,x_3) \cdot h(x_2|s_1,s_2,s;x_1,x) \ .$$

Then (3.8) follows from an interchange of integrals, (3.9), and finally, an application of formula (2.10).

It is important to note the symmetry between formulas (2.7) and (3.5) and between (2.9) and (3.6). Because of this symmetry, procedures I and II can be extended in obvious ways to include the computation of moments of τ. An alternative approach is to use recursive formula (2.8) and compute the moments directly from (3.1).

4. Computing the "fundamental probabilities".

Section two defines a fundamental probability to be either of the following two types of probabilities:

(i) $P_i(s_1,s_2;x_1)$, for some $i=0,1,\ldots,K$,

where $0 \leq s_1 < s_2 \leq \infty$ and (s_1,s_2) contains no critical times.

(ii) $Q_i(s_1,s_2;x_1,x_2)$, for some $i=0,1,\ldots,K$,

where $0 \leq s_1 < s_2 < \infty$ and (s_1,s_2) contains no critical times.

For $s_2 < \infty$, we may use the formula

$$P_i(s_1,s_2;x_1) = \int_{-\infty}^{\infty} Q_i(s_1,s_2;x_1,x_2)g(x_2|s_1,s_2;x_1)dx_2 \ . \tag{4.1}$$

It remains to show how (i) is computed when $s_2 = \infty$, and how (ii) is computed in general.

Computing $P_i(s,\infty;x)$:

We assume that the interval (s,∞) contains no critical times and the point (s,x) is an accessible point. The point (s,x) may or may not have a non-vertical boundary line below (above) it which extends to infinity. For this reason, there are three cases to be considered:

(a) The two line case - at least one boundary line above and one below the point (s,x).

(b) The one line case - at least one boundary line above or at least one below the point (s,x), but not both.

(c) The zero line case - no boundary lines above or below the point (s,x).

If the process $X(t)$ passes through (s,x) and contacts the boundary after time s, its first such contact will be with the boundary line below (s,x) with largest slope or the one above with smallest slope. If we are in case (a) and all of the boundary lines below (s,x) diverge away from all the boundary lines above (s,x) or if we are in case (b) or (c), the test can not be closed for all values of μ. This is a consequence of the third characterization of the class of boundary tests which are closed for all μ (given in section 1).

While case (c) is trivial, cases (a) and (b) are treated by using Theorem 1 and Theorem 2, respectively.

Theorem 1. (T.W. Anderson [1]): *Let* $X(t) \sim N(\mu t, \sigma^2 t)$ *be a Wiener process. Let* $\gamma_1 + \delta_1 t$ *and* $\gamma_2 + \delta_2 t$ (*for* $t \geq 0$) *be two parallel or diverging lines with* $\gamma_1 < 0 < \gamma_2$ *and* $\delta_1 \leq \delta_2$. *The probability that* $X(t)$ *makes contact with the lower line* $\gamma_1 + \delta_1 t$ *before the upper line* $\gamma_2 + \delta_2 t$ *is given by:*

$$\sum_{r=1}^{\infty} \left\{ e^{-\frac{2}{\sigma^2}[r\gamma_1-(r-1)\gamma_2][r(\delta_1-\mu)-(r-1)(\delta_2-\mu)]} - e^{-\frac{2}{\sigma^2}[r^2\{\gamma_1(\delta_1-\mu)+\gamma_2(\delta_2-\mu)\}-r(r-1)\gamma_1(\delta_2-\mu)-r(r+1)\gamma_2(\delta_1-\mu)]} \right\},$$

for $\delta_1 \leq 0$, $\delta_1 < \delta_2$;

$$1 - \sum_{r=1}^{\infty} \left\{ e^{-\frac{2}{\sigma^2}[(r-1)\gamma_1-r\gamma_2][(r-1)(\delta_1-\mu)-r(\delta_2-\mu)]} - e^{-\frac{2}{\sigma^2}[r^2\{\gamma_1(\delta_1-\mu)+\gamma_2(\delta_2-\mu)\}-r(r+1)\gamma_1(\delta_2-\mu)-r(r-1)\gamma_2(\delta_1-\mu)]} \right\},$$

If the process $X(t)$ passes through (s,x) and contacts the boundary after time s, its first such contact will be with the boundary line below (s,x) with larger slope or the one above with smallest slope. If we are in case (a) all of the boundary lines below (s,x) diverge away from all the boundary lines above (s,x) or if we are in case (b) or (c), the test must be closed for all values of μ. This is a consequence of the third characterization of the class of boundary tests which are closed for all μ (given in section 1).

While case (c) is trivial, cases (a) and (b) are treated by using Theorem 1 and Theorem 2, respectively.

<u>Theorem 1</u> (T.W. Anderson [1]): <u>Let $X(t) \sim N(\mu t, \sigma^2 t)$ be a Wiener process. Let $\gamma_1 + \delta_1 t$ and $\gamma_2 + \delta_2 t$ (for $t \geq 0$) be two parallel or diverging lines with $\gamma_1 < 0 < \gamma_2$ and $\delta_1 \leq \delta_2$. The probability that $X(t)$ makes contact with the lower line $\gamma_1 + \delta_1 t$ before the upper line $\gamma_2 + \delta_2 t$ is given by:</u>

$$\sum_{r=1}^{\infty}\left\{ e^{-\frac{2}{\sigma^2}[r\gamma_1-(r-1)\gamma_2][r(\delta_1-\mu)-(r-1)(\delta_2-\mu)]} - e^{-\frac{2}{\sigma^2}[r^2(\gamma_1(\delta_1-\mu)-\gamma_2(\delta_2-\mu))-r(r-1)\gamma_1(\delta_2-\mu)-r(r+1)\gamma_2(\delta_1-\mu)]}\right\}$$

for $\delta_1 < 0$, $\delta_1 < \delta_2$

$$1 - \sum_{r=1}^{\infty}\left\{ e^{-\frac{2}{\sigma^2}[r\gamma_2-(r-1)\gamma_1][(r-1)(\delta_1-\mu)-r(\delta_2-\mu)]} - e^{-\frac{2}{\sigma^2}[r^2(\gamma_1(\delta_1-\mu)+\gamma_2(\delta_2-\mu))-r(r+1)\gamma_1(\delta_2-\mu)-r(r-1)\gamma_2(\delta_1-\mu)]}\right\}$$

for $0 \le \delta_1 < \delta_2$;

$$\left(e^{-\frac{2}{\sigma^2}\gamma_2(\delta_1-\mu)}-1\right)\left(e^{-\frac{2}{\sigma^2}(\gamma_2-\gamma_1)(\delta_1-\mu)}-1\right)^{-1}, \text{ for } \delta_1 = \delta_2 \neq \mu ;$$

and

$$\gamma_2(\gamma_2-\gamma_1)^{-1}, \quad \text{for } \delta_1 = \delta_2 = \mu .$$

Theorem 2. (J.L. Doob [**2**]): *Let* $X(t) \sim N(\mu t, \sigma^2 t)$ *be a Wiener process. Let* $\gamma + \delta t$ (*for* $t \geq 0$) *be a straight line with* $\gamma \neq 0$. *The probability that* $X(t)$ *contacts the line is given by:*

$$\min\left(1, e^{-\frac{2}{\sigma^2}\gamma(\delta-\mu)}\right) .$$

Computing $Q_i(s_1,s_2;x_1,x_2)$: We assume the interval (s_1,s_2) contains no critical times and the point (s_1,x_1) is an accessible point.

Define

(4.2) $R_i(s_1,s_2;x_1,x_2) \equiv P[(t,X(t))\epsilon B_i$ before $\overline{B}_i$ for some time $t\epsilon(s_1,s_2)|X(s_j)=x_j,\ j=1,2]$,

for $i=1,\ldots,K$, and

(4.3) $R_0(s_1,s_2;x_1,x_2) \equiv P[(t,X(t))\epsilon B_0$ for all times $t\epsilon(s_1,s_2)|X(s_j)=x_j, j=1,2]$.

While the values of $Q_i(s_1,s_2;x_1,x_2)$ is affected by the existence or non-existence of vertical boundary lines in the line $t = s_2$, the value of $R_i(s_1,s_2;x_1,x_2)$ is not. This makes the computation of the R's more straightforward than the computation of the Q's. We shall develop methods for computing the R's and use the following formula to compute the Q's.

$$(4.4) \quad Q_i(s_1,s_2;x_1,x_2) = (1-\delta_{i0})R_i(s_1,s_2;x_1,x_2)+\delta_{ij}R_0(s_1,s_2;x_1,x_2) ,$$

for $i=0,1,\ldots,K$, where $(s_2,x_2)\epsilon B_j$, and where $\delta_{\alpha\beta}$ is the Kronecker delta $(\alpha,\beta = 0,1,\ldots,K)$. Note that $(s_2,x_2)\epsilon B_j$ for some $j=1,\ldots,K$ or for $j=0$. (4.4) may be verified by considering each of the various cases encompassed by the formula.

In computing $R_i(s_1,s_2;x_1,x_2)$, it is clear that we have three cases to consider:

(a) The two line case - at least one non-vertical boundary line below and at least one above the point (s_1,x_1) which extend to the right and do not cross before time s_2.

(b) The one line case - at least one non-vertical boundary line below the point (s_1,x_1) or at least one above, but not both.

(c) The zero line case - no boundary lines between time s_1 and time s_2.

Again, the zero line case is trivial. Cases (a) and (b) are treated by using theorems 3 and 4, respectively.

Theorem 3. (T.W. Anderson [1]): Let $X(t) \sim N(\mu t, \sigma^2 t)$ be a Wiener process. Let $T > 0$ be a fixed time. Let $\gamma_1 + \delta_1 t$ and $\gamma_2 + \delta_2 t$ (for $t \geq 0$) be two lines which do not intersect before $t = T$ with $\gamma_1 < 0 < \gamma_2$. Thus $\gamma_1 + \delta_1 T \leq \gamma_2 + \delta_2 T$. The conditional probability that $X(t)$ makes contact with the lower line $\gamma_1 + \delta_1 t$ before contacting the upper line $\gamma_2 + \delta_2 t$ and before time $t=T$ given $X(T) = x$ is given by:

$$\sum_{r=1}^{\infty} \left\{ e^{-\frac{2}{\sigma^2 T}[r\gamma_1-(r-1)\gamma_2][r(\gamma_1+\delta_1 T-x)-(r-1)(\gamma_2+\delta_2 T-x)]} - e^{-\frac{2}{\sigma^2 T}[r^2\{\gamma_1(\gamma_1+\delta_1 T-x)+\gamma_2(\gamma_2+\delta_2 T-x)\}-r(r-1)\gamma_1(\gamma_2+\delta_2 T-x)-r(r+1)\gamma_2(\gamma_1+\delta_1 T-x)]} \right\},$$

for $x \geq \gamma_1 + \delta_1 T$ and $\gamma_1 + \delta_1 T < \gamma_2 + \delta_2 T$;

$$1 - \sum_{r=1}^{\infty} \left\{ e^{-\frac{2}{\sigma^2 T}[(r-1)\gamma_1-r\gamma_2][(r-1)(\gamma_1+\delta_1 T-x)-r(\gamma_2+\delta_2 T-x)]} - e^{-\frac{2}{\sigma^2 T}[r^2\{\gamma_1(\gamma_1+\delta_1 T-x)+\gamma_2(\gamma_2+\delta_2 T-x)\}-r(r+1)\gamma_1(\gamma_2+\delta_2 T-x)-r(r-1)\gamma_2(\gamma_1+\delta_1 T-x)]} \right\},$$

for $x \leq \gamma_1+\delta_1 T < \gamma_2+\delta_2 T$;

$$\left(e^{-\frac{2}{\sigma^2 T}\gamma_2(\gamma_1+\delta_1 T-x)} - 1\right)\left(e^{-\frac{2}{\sigma^2 T}(\gamma_2-\gamma_1)(\gamma_1+\delta_1 T-x)} - 1\right)^{-1},$$

for $x \neq \gamma_1+\delta_1 T = \gamma_2+\delta_2 T$;

and

$\gamma_2(\gamma_2-\gamma_1)^{-1}$, for $x = \gamma_1+\delta_1 T = \gamma_2+\delta_2 T$.

Theorem 4. Let $X(t) \sim N(\mu t,\sigma^2 t)$ be a Wiener process. Let $\gamma_1+\delta_1 t$ be any straight line $(t \geq 0)$ with $\gamma \neq 0$. Let $T > 0$ be a fixed time. The conditional probability that $X(t)$ contacts the line before time $t = T$ given $X(T) = x$ is given by:

$$\min\left(1, e^{-\frac{2}{\sigma^2 T}\gamma(\gamma+\delta T-x)}\right) .$$

Theorem 4 is proven by using the same technique that Anderson used in proving theorem 3.

Some of the indicated computations in this paper, such as certain integrations, can be carried further. In some cases, this is desirable. However, this matter involves certain digressions which seem inappropriate in a paper of this type. For details, see [3].

When μ has a prior distribution, it is possible to extend the methods above in order to compute "global" acceptance probabilities and "global" moments of τ. The extension is easily achieved by modifying procedure II.

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